Relational relationships

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The language of *relations* in mathematics gives a formal way to talk about various relationships between things, such as the "is greater than or equal to" relationship between numbers, or the "is a child of" relationship between people, or the "is adjacent to" relationship between countries. This language allows you to build new relations out of existing ones and to highlight special properties of relations (for example, the symmetry of a relationship like siblinghood).

One exciting possibility is applying this relational language to the relations in human *relationships*. In this article, I show how many familiar properties of relationships can be expressed succinctly in standard relational terms, and moreover how the apparatus of relational language helps us discover a few unfamiliar surprises—such as the beautifully accurate idea that your spouse is your *self-in-law*.

1 The language of relations

Terms

This section introduces the mathematical underpinnings, including definitions and notations.

If you have a set of things B and a set of things A, the **cartesian product** of B and A is the set of all possible ordered pairs where the first item is from B and the second item is from A. The cartesian product is denoted $B \times A$.

$$B \times A \equiv \{\langle b, a\rangle : b \in b,\ a \in A\}$$

A (binary) **relation** between a set of things B and a set of things A is any subset of the cartesian product $B \times A$. Intuitively, a relation is the set of things that you consider to be related to each other. For example, the less-than relation $\leq$ on positive integers is formally the set of all pairs of integers where the first item is less than or equal to the second:

$$\leq = \ \{\langle m, n\rangle : m \leq n\} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \ldots\}$$

Well-known examples of relations in mathematics include $\leq$ and $\neq$. By extension from these familiar examples, if R is any relation, you can use the notation bRa to mean that b is related to a, or formally that $\langle b, a\rangle \in R$.

If you have any relation $R \subseteq B \times A$, you can form the **opposite relation** $R^\circ \subseteq A \times B$. The opposite relation is formally defined as

$$R^\circ \equiv \{\langle a, b\rangle \subseteq A \times B : \langle b, a\rangle \in R\}$$

Intuitively, R° is the reverse of the relationship R:

- When R is the *is-parent-of* relation, R° is the *is-child-of* relation.
- When R is the $\leq$ relation, then R° is the $\geq$ relation.
- When R is the *lives-in-place* relation between people and geographic locations, then R° is the *contains-inhabitant* relation between geographic locations and people.

A relation can either be between two different sets $B \times A$, or the same set $A \times A$. For any set of things A, there is a special **identity relation** $id_A \subseteq A \times A$, where each item in A is related to itself and nothing else. In some contexts, it's useful to think of the identity relation as meaning "is the same as" for the set A.

Relations can be **contained in** one another: if R and S are two relations on $B \times A$, we use the standard notation $R \subseteq S$ to mean that R is a subset of S. In the context of relations, this means that R is a more restrictive form of the relationship S. For example, the relation $<$ is more restrictive than the relation $\leq$.

Relations can also be **intersected with** one another: if R and S are two relations on $B \times A$, we use the standard notation $R \cap S$ to refer to the intersection of the two relations as sets. In the context of relations, $R \cap S$ is the relation "bRa and also bSa".

Relations can be **composed with** one another: if S is a relation on $C \times B$ and R is a relation on $B \times A$, then $S \cdot R$ is a relation on $C \times A$ defined by

$$S \cdot R \equiv \{\langle c, a\rangle : cSb \text{ and } bRa\}$$

For example, if S is the relation *is-parent-of* and R is the relation *is-friend-of*, then $S \cdot R$ is the relationship *is-parent-of-friend-of* and $R \cdot S$ is *is-friend-of-parent-of*. You can prove to yourself that identity relations cancel when you compose them: $id_B \cdot R = R \cdot id_A = R$.

Conditions

Some relations have special properties. This section lists the names and definitions of some of the important properties. For the purposes of this section, all relations are assumed to be on the *same* set: $R \subseteq A \times A$.

If you have seen terms like transitivity before, one surprise is how the relational algebra operators $^\circ$, $\subseteq$, $\cap$, and $\cdot$ allow you to describe familiar relational terms in a succinct and point-free manner, e.g. reflexivity ($id_A \subseteq R$), symmetry ($R = R^\circ$), and transitivity ($R \subseteq R \cdot R$).

Reflexive A relation on $A \times A$ is reflexive if every item is related to itself. For example, relations like $=$ and $\leq$ and "has compatible blood type with" are reflexive, while $\neq$ and $<$ and "is parent of" are not. Formally, a relation on $A \times A$ is reflexive if $id_A \subseteq R$.

Coreflexive A relation on $A \times A$ is coreflexive if $R \subseteq id_A$ (this is the dual of the condition for being reflexive). The coreflexive relations on $A \times A$ correspond to the subsets of A: if S is any subset of A, then $\{\langle a, a\rangle \in A \times A : a \in S\}$ is a coreflexive relation, and all coreflexive relations have this form.

Symmetric A relation on $A \times A$ is symmetric if whenever bRa, it's also true that aRb. For example, *is-sibling-of* is symmetric while *is-parent-of* is not. Formally, a relation on $A \times A$ is symmetric if $R = R^{\circ}$.

Transitive A relation on $A \times A$ is transitive if whenever cRb and bRa, you can always conclude that cRa. For example, relations like $\leq$ are transitive because whenever $c \leq b$ and $b \leq a$, you can always conclude that $c \leq a$. On the other hand, relations like $\neq$ are not transitive: if you have $c \neq b$ and $b \neq a$, you don't necessarily know that $c \neq a$. Formally, a relation on $A \times A$ is transitive if $R \subseteq R \cdot R$.

Irreflexive A relation on $A \times A$ is irreflexive if no item is ever related to itself. For example, the *is-parent-of* relation is irreflexive provided no one is their own parent. Formally, a relation $A \times A$ is irreflexive if $R \cap id_A = \varnothing$.

Antisymmetric A relation on $A \times A$ is antisymmetric if the only time we have both bRa and aRb is when $a = b$. For example, the $\leq$ relation is antisymmetric, while the *is-sibling-of* relation is not. Formally, a relation is antisymmetric when $R \cap R^{\circ} \subseteq id_A$

Equivalence A relation on $A \times A$ is an equivalence relation if it is reflexive, symmetric, and transitive. Intuitively, equivalence relations capture a sense in which items can be "the same as" one another. For example, "has the same height" is an equivalence relation on buildings.

2 Human relationships

In this section, we apply the language of relations to human relationships. These are a special case of relationships on the set of all people.

The Spousal relation

The spousal relation $\mathscr{U}$ is the relationship *is-spouse-of.* ("$\mathscr{U}$" can be a mnemonic for union, uxorial, etc.) Depending on the rules and customs of your society, the spousal operator may have different required or observed properties.

This section lists some of those properties along with their formal definition in relational terms. As a fun exercise, you may try to determine for yourself a formal relational definition of each concept before looking at mine. For that purpose, the relationship properties I will define are:

> Symmetry, irreflexivity, monogamy (each person has at most one spouse), one-sided monogamy (each person in a set E has at most one spouse; other people are unrestricted), celibacy (each person in a set E has no

spouse; other people are unrestricted), transitivity (if c is b's spouse and b is a's spouse, then c is a's spouse), in-grouping (people are divided into distinct groups, and people must marry within the same group), and out-grouping (people are divided into distinct groups, and people must marry into a different group.)

Definitions follow.

Symmetry Usually, if a is b's spouse, then b is a's spouse. $\mathscr{U}^\circ = \mathscr{U}$.

Irreflexivity Usually, you cannot be your own spouse. $\mathscr{U} \cap id = \varnothing$.

Monogamy The monogamy condition—the condition that each person have at most one spouse—is neatly expressed as the requirement that $\mathscr{U}^\circ \cdot \mathscr{U} \subseteq id$ ("my spouse's spouse can only be me.")[1]

One-sided monogamy Suppose only a subset E of people is required to be monogamous while the rest are unrestricted. We can express this condition succinctly as $\mathscr{U}^\circ \cdot \widehat{E} \cdot \mathscr{U} \subseteq \widehat{E}$, where here $\widehat{E}$ refers to the coreflexive relation corresponding to the subset E. (Intuitively, this definition reads: If your spouse is someone from E, their spouse must be you.) Monogamy in the previous definition implies every kind of one-sided monogamy.

Celibacy/Nilgamy Suppose a subset of people E is required not to have any spouse at all. This condition is $\mathscr{U}^\circ \cdot \widehat{E} = \widehat{E} \cdot \mathscr{U} = \varnothing$. ("No person in E has a spouse, and no one has a person in E as a spouse.").

Transitivity The spousal relation will be transitive if all marriages are "mutual" in a certain sense; for example, in order to be transitive, a symmetric spousal relation must be reflexive and e.g. "altergamous" in the sense that each person has their own self and at most one other person as a spouse. (It is sometimes useful to stretch the everyday definition of spousehood such that each person is considered to be their own spouse.) The spousal relation might not be transitive if concubine-like polygamous relationships exist, in which case there may be two people married to the same person but not to each other.

In-grouping/Endogamy Suppose society is partitioned into particular demographic groups and one can only have a spouse from the same group as one's self. (Examples include conditions based on nationality, race, class, or caste.) Let us capture the notion of these groups as a relation G "belongs to the same in-group as". (Presumably, G is reflexive and symmetric, like an equivalence relation, but possibly not transitive.) The in-grouping condition is simply $\mathscr{U} \subseteq G$ ("being spouses is a strictly stronger condition than belonging to the same in-group").

[1] This definition of monogamy looks reminiscent of the definition of a unitary transformations in linear algebra. And there's some conceptual similarity in the oneness of mono-gamy and the oneness of unit-ary transformations. Because I think that's neat, I write the definition as $\mathscr{U}^\circ \cdot \mathscr{U}$ instead of the simpler equivalent definition $\mathscr{U} \cdot \mathscr{U}$ even though the spousal relation is typically symmetric.

Out-grouping/Exogamy Suppose society is partitioned into particular groups and one can only have a spouse from a *different* group. (Examples include conditions based on kinship[2] or gender.) If G again denotes the relation "belongs to the same in-group as", then the out-grouping condition is simply $\mathcal{U} \cap G = \varnothing$.

In-laws

We can form in-law relations by composing with $\mathscr{U}$. For example, if R denotes the *is-parent-of* relation, then $R \cdot \mathscr{U}$ denotes the relation relating people to the parents of their spouses—what we commonly refer to as the relation *parent-in-law*.

If R is instead the sibling relation, then this same construction $R \cdot \mathscr{U}$ yields the "spouse's sibling" relation—*sibling-in-law*.

In English, there are two kinds of sibling-in-law: your sibling's spouse and your spouse's sibling. These are generally different because $\mathscr{U} \cdot \mathsf{Sibling} \neq \mathsf{Sibling} \cdot \mathscr{U}$. In general, we can fill out a table with different kinship relations R, and the everyday meaning of $R \cdot \mathscr{U}$ and $\mathscr{U} \cdot R$.

We can fill out a table where we list kinship relations R and the corresponding constructs $R \cdot \mathscr{U}$ and $\mathscr{U} \cdot R$. The entries will reveal how the concepts and terminology for $\mathscr{U} \cdot R$ and $R \cdot \mathscr{U}$ differ.

Kinship relation R	$R \cdot \mathscr{U}$	$\mathscr{U} \cdot R$
Parent	parent-in-law	parent's spouse[3]
Sibling	sibling-in-law (spouse's sibling)	sibling-in-law (sibling's spouse)
Child	step-child or child	child-in-law
Self ($R = id$)	spouse — *self-in-law* (!)	[same]
Spouse ($R = \mathscr{U}^\circ$)	self (monogamous) or metamour (polygamous)	[same]
Friend(?)	friend-in-law(?) (spouse's friend)	friend-in-law(?) (friend's spouse)

This table suggests thought-provoking new terminology and reveals surprising relationships between existing terms.

Consider, for example, that if your spouse's parent is your parent-in-law and your spouse's sibling is your sibling-in-law, then by extension, your spouse's own self ought to be your **self-in-law** (!). Note that in many communities and for many practical and legal purposes—e.g. taxation, spousal testimonial privilege, medical visitation—spouses can indeed be considered a legal extension of one's own self.

Or consider the potentially novel idea that the concepts child-in-law and step-child are dual to one another: one is your child's spouse and the other is your spouse's child—and *both* are ways of adding a child into your family via a marriage. In one case, it is *your* marriage which introduces the new child; in the other, it is your *child*'s marriage[4].

[2] i.e. prohibitions against incest

[4] Actually, this duality is not special to step-children and children-in-law: analogous reasoning applies to all other pairs of relations $R \cdot \mathscr{U}$ and $\mathscr{U} \cdot R$ in a row of the table. Both relations must be ways of adding

Or if we extend our relations R to include non-kinship relations, we can coin humorous new terms such as **friend-in-law** for our spouse's friends (or friends' spouses), or perhaps even **house-in-law** for our spouse's house, **hometown-in-law** for our spouse's hometown, or **vocation-in-law** for our spouse's vocation (especially a spousal vocation that one is personally invested in).

For the less-legally-inclined, all results in this article apply equally well to other interpersonal relationships besides marriage.

3 A side note on cousins

The terminology for first, second, and nth cousins is sometimes bewildering, even before the term "k times removed" is introduced. The relational apparatus we have developed may make it easier to express these consanguinity relations.

- Cousins are people with whom we share a common ancestor.
- Removal occurs when cousins belong to different generations.
- The parental relation P relates people to their parents: bPa means that b is the parent of a.
- Interestingly, we can express the sibling relation as $P^\circ \cdot P$.
- Grandparents are $P^2 = P \cdot P$, and first cousins are essentially $(P^\circ \cdot P^\circ) \cdot (P \cdot P)$—they're people with whom you share grandparents.
- More generally, nth cousins are people for whom your closest common ancestor is $n+1$ generations above you. We may as well write that your nth cousin is $(P^\circ)^{n+1}P^{n+1}$. By extreme extension of this reasoning, your siblings are your 0th cousins, and you are your own -1th cousin.
- There's a slight bug in this initial definition, as it includes too many people—you, your siblings, your cousins, and your second cousins all share great-grandparents, whereas only second cousins share nothing else. If we want the definition of cousins to be proper, we should actually define nth cousins to be

$$\begin{aligned} C_{-1} &= id && \{\text{selfhood}\} \\ C_0 &= P^\circ P - id && \{\text{siblings}\} \\ C_{n+1} &= P^\circ C_n P - id && \{(n+1)\text{th cousins}\} \end{aligned}$$

a new R to your family, either by you marrying someone, or by your R marrying someone. However, the English terminology usually exposes the similarity, whereas child-in-law and step-child does not—hence I find the case $R = \mathsf{child}$ to be especially novel.

- "Removal" occurs whenever you and another person have a common ancestor but belong to different generations. For example, you and your cousin's child have the same common ancestor as you and your cousin, but you are one generation apart. Hence you and your cousin's child are first cousins, one time removed.

  In general, the relation "nth cousin, k times removed" refers to

  $$C_n P^k \cup P^k C_n \cup C_n (P^\circ)^k \cup (P^\circ)^k C_n.$$

  (i.e. k generations up or down from your nth cousins.)

- Amusingly, by extension of common usage to include siblings as "0th cousins", your parents and their siblings are your "0th cousins, once removed", as are your children and your siblings' children. Similarly, your grand-parents and their siblings are your "0th cousins, twice removed", as are your grandchildren and your siblings' grandchildren.