1 A matrix with all row and column sums equal to k

Suppose A is an $n \times n$ matrix of zeroes and ones, and that $A^2 = J$, the matrix of all ones. Show that:

1. Every row and column of A adds up to the same number, k.
2. The number k satisfies $k^2 = n$, where n is the size of the matrix.
3. The matrix A has exactly k ones along the diagonal.

Proof. The first trick is to note that matrix multiplication by J actually computes the column/row sums of a matrix: For any order-n matrix M, let $r_1, \ldots, r_n$ be its row sums. Note that by matrix multiplication rules, MJ is the matrix of row sums:

$$MJ = \begin{bmatrix} r_1 & r_1 & \ldots & r_1 \\ r_2 & r_2 & \ldots & r_2 \\ \vdots & \vdots & & \vdots \\ r_n & r_n & \ldots & r_n \end{bmatrix}$$

and similarly JM is the matrix of column sums.

1. *The row/column sums are* k. We know that $A^2 = J$. Hence:

   $A^3 = AJ = JA$, when we multiply by A.

   But AJ computes the row sums of A, and JA computes the column sums. If they're equal, it means that each row and column of A adds up to the same number— call it k. (The sum k must be positive because A consists of zeroes and ones, and A is not the zero matrix.)

   Hence we can write $AJ = JA = kJ$ (the matrix where each entry is k.)

2. *The number* $k^2 = n$. We can show that $J^2 = nJ$, because the sum of every row/column of J is just n. But because $AJ = kJ$, we also know that:

   $J^2 = JA^2 = JAA = (JA)A = (kJ)A = k(JA) = k(kJ) = k^2 J$

   Hence $J^2 = nJ$ and also $J^2 = k^2 J$

   In other words, $k^2 = n$.

3. *A has exactly k ones along the diagonal.* We can show that there are exactly k ones along the diagonal using some facts about eigenvalues.

   If A consists of zeroes and ones, then the number of ones along the diagonal is equal to the sum of the values along the diagonal. This is the *trace* of matrix A. Hence we need to show that the trace of A is equal to k. The trace of a matrix is also equal to the sum of its (generalized) eigenvalues, so it suffices to show that A has exactly one (nonzero) generalized eigenvalue, namely k.

   We already know that $AJ = kJ$. Looking at a single column, it follows that the vector of all ones is an eigenvector of A with eigenvalue k.

   Combining $AJ = kJ$ with $A^2 = J$, we find that $A^3 = kA^2$.

   The generalized eigenvector formula is $(A - \lambda I)^n v = 0$. In particular, for the eigenvalue $\lambda = 0$, this becomes $A^n v = 0$. Because $A^3 = kA^2$, we can prove by induction that $A^n = k^{n-2} A^2$, so $A^n v = 0$ just if $A^2 v = 0$.

   Hence v is a generalized eigenvector of A with eigenvalue 0 just if $A^2 v = Jv = 0$. But Jv returns a vector where each entry is the sum of all the entries in v; hence $Jv = 0$ just if the entries of v sum to zero. The set of all such vectors has dimension $n - 1$ (it's the set of vectors perpendicular to $[1, 1, \ldots, 1]$) and so 0 is a generalized eigenvalue of A with multiplicity $n - 1$.

But an $n \times n$ matrix has exactly n generalized eigenvalues. Hence k must be one of them, and 0 (counting multiplicities) must be the rest.

It follows that the trace of A is equal to $k = k + 0 + 0 + \ldots + 0$, which proves that A has exactly k ones along the diagonal.

Generalizations As an aside, none of these results really depend on A being a matrix of zeroes and ones. In fact, using the same reasoning as above, we can prove the following general result for any $M^r = J$:

Let $r \geq 1$ be an integer and suppose M is *any* $n \times n$ matrix with $M^r = J$. Then each row and each column of M will sum to $k \equiv n^{1/r}$; the trace of M will be equal to k (not any power; still just k itself); and if M consists of zeroes and ones, it will have exactly k ones along the diagonal.

Concrete construction If you want a concrete construction of matrices A with the above property: define the matrix $M[k, r, a]$ to be an order k^r matrix with

$$M[k, r, a]_{i,j} = [1 \text{ if } (0 \leq ik^a + j < k^a) \pmod{k^r};\ 0 \text{ otherwise }]$$

This complicated formula is better explained by visual example:

$$M[k = 2; r = 3; a = 1] = \begin{bmatrix} 1 & 1 & & & & & & \\ & & 1 & 1 & & & & \\ & & & & 1 & 1 & & \\ & & & & & & 1 & 1 \\ 1 & 1 & & & & & & \\ & & 1 & 1 & & & & \\ & & & & 1 & 1 & & \\ & & & & & & 1 & 1 \end{bmatrix}$$

where the length of the runs is determined by k^a, and the size of the matrix is determined by k^r.

Then I claim without proof that that $A \equiv M[k, r, 1]$ is an order-k^r matrix with $A^r = J$.

More generally, I expect $M[k, r, a] \cdot M[k, r, b] = M[k, r, a + b]$. (In particular, $M[k, r, 0]$ is the order-k^r identity matrix.)

Problem sources

https://math.stackexchange.com/questions/2516152/a-has-all-line-sums-equal-to-a-positive-number-k/2516281#2516281

https://math.stackexchange.com/questions/2517206/a-regular-connected-graph-has-k-loops/2519392?noredirect=1

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